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ON ANTI-SYMMETRIC WAVES IN AN UNBOUNDED PIEZOELECTRIC PLATE WITH AXISYMMETRIC ELECTRODES

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Abstract—In this paper we consider an unbounded piezoelectric plate with electrodes at its surfaces, which are axisymmetric with respect to an axis normal to the plate. The electromechanical waves in this plate are assumed to have an electric potential which is symmetric with respect to the middle plane of the plate. The electric field outside the plate will appear to be important. This external electric field is investigated, particularly in connection with the resonances of the plate.

1. INTRODUCTION

We consider an unbounded piezoelectric plate. The polarization direction of the material of the plate is perpendicular with respect to the surfaces of the plate. At the upper surface of the plate there are two electrodes: one circular electrode and one ring-shaped electrode, which encircles the circular electrode concentrically. At the lower surface there are also two electrodes; the upper electrodes and the lower ones are symmetric with respect to the middle plane of the plate. The plate with electrodes can be considered as an electric four-pole.

It will appear that, within the linear theory, each electromechanical wave that can occur in the plate with electrodes, can be written as the sum of a symmetric wave and an anti-symmetric wave. In applications the symmetric waves are the most important ones. However, if the plate with electrodes is circuited in a network, the waves generated in the plate are purely symmetric only if the network is symmetric in a certain sense. For instance, if the two lower electrodes are connected to earth, then the waves in the plate are not purely symmetric. The symmetric waves have been treated for instance in[1, 5, 6]. In this paper we will investigate the anti-symmetric waves.

The plate considered in this paper will have a large relative dielectric constant. In many papers treating piezoelectric bodies with a large dielectric constant, it is common practice to set, at the surface of the body, the limit value from outside the body of the normal component of the electric displacement equal to zero. Then the electric field outside the body is left out of consideration. Especially when a piezoelectric body of a more complicated shape is treated, this common practice is necessary in order to simplify the equations.

For our plate we will give results obtained from computations at which all equations inside and outside the plate are satisfied, and we will give results obtained from computations at which the above-mentioned common practice has been applied. It appears that the results are in good agreement, except for some resonances: The plate with electrodes has a number of resonances according to the exact computations and only a part of this number of resonances follows also from the computations at which the above-mentioned common practice is applied.

2. GEOMETRIC CONFIGURATION AND BASIC EQUATIONS

We consider an unbounded piezoelectric plate in vacuum. In polar coordinates (r, φ, z) the faces of the plate are at $z = \pm h$; $h = 1$. The polarization direction of the ceramic is parallel to the z-axis and the material is homogeneous throughout the plate. The plate is provided with four electrodes, numbered 1, 2, 3 and 4. The electrodes are symmetric with respect to the plane $z = 0$ and are axisymmetric with respect to the z-axis. The electrodes 1 and 3 are circular with radius a_1. The electrodes 2 and 4 are rings with inner radius a_2 and outer radius a_3 (see Fig. 1). The following conditions are assumed to be satisfied: (i) The electrodes are infinitely thin, so that they are not able to induce any mechanical effect, and are perfect conductors. (ii) The mechanical stresses and strains and the electromagnetic field are small, so that a linear relation between those quantities holds. (iii) All field-quantities have harmonic time dependence $e^{-i\omega t}$. The angular

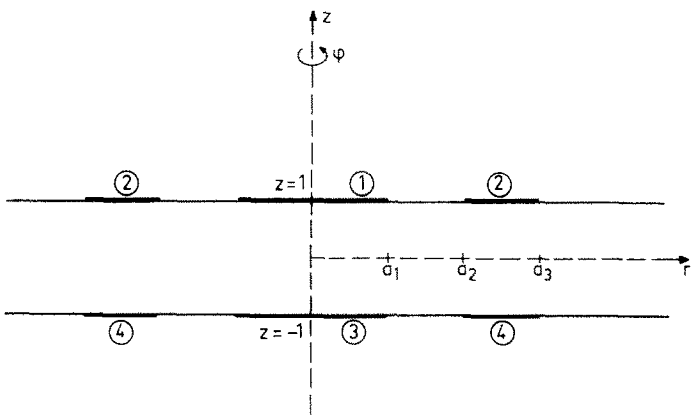

Fig. 1. The unbounded plate with electrodes.

frequency ω is not too large, so that the quasi-static Maxwell equations may be applied. We assume the field-quantities to have the following symmetry. Let U (resp. W) denote the complex amplitude of the particle-displacement in the r-direction (resp. z-direction) and let V denote the complex amplitude of the electric potential. Then U, W and V are functions of r and z, but are independent of φ. Moreover, the particle-displacement in the φ-direction is assumed to vanish identically.

We will state equations valid in $|z| < 1$, equations valid in $|z| > 1$ and boundary- and transition conditions valid at $z = \pm 1$ respectively. Inside the plate we use the following equations. The constitutive equations for piezoelectric material are usually given in a cartesian coordinate system (x_1, x_2, x_3). The system is chosen such that the x_3-axis coincides with the z-axis. Then the equations read:

$$T^{ij} = c^{ijkl} S_{kl} - e^{kij} E_k$$

$$D^i = e^{ikl} S_{kl} + \epsilon^{ik} E_k \qquad i,j = 1,2,3 \tag{1}$$

These equations give a linear relation between the components of four tensors (vectors): T denotes the stress tensor, S the strain tensor, E the electric field vector and D the electric displacement vector. The material constants c^{ijkl}, e^{kij} and ϵ^{ik} are respectively elastic, piezoelectric and dielectric coefficients.
The material constants have the usual symmetry in the indices as given for instance in[6]. The indices i, j, k and l all run over the values 1, 2 and 3 and the summation convention for repeated indices is applied. A tensor transformation of (1) to the polar coordinate system and use of the symmetry introduced above gives:

$$\left.\begin{aligned}
T^{rr} &= c^{1111} S_{rr} + \frac{c^{1122}}{r^2} S_{\varphi\varphi} + c^{1133} S_{zz} - e^{311} E_z,\\
T^{\varphi\varphi} &= \frac{c^{1122}}{r^2} S_{rr} + \frac{c^{1111}}{r^4} S_{\varphi\varphi} + \frac{c^{1133}}{r^2} S_{zz} - \frac{e^{311}}{r^2} E_z,\\
T^{zz} &= c^{1133} S_{rr} + \frac{c^{1133}}{r^2} S_{\varphi\varphi} + c^{3333} S_{zz} - e^{333} E_z,\\
T^{rz} &= 2c^{1313} S_{rz} - e^{113} E_r,
\end{aligned}\right\} \tag{2}$$

$$\left.\begin{aligned}
D^r &= 2e^{113} S_{rz} + \epsilon^{11} E_r,\\
D^z &= e^{311} S_{rr} + \frac{e^{311}}{r^2} S_{\varphi\varphi} + e^{333} S_{zz} + \epsilon^{33} E_z,
\end{aligned}\right\} \tag{3}$$

while the stresses $T^{r\varphi}$ and $T^{\varphi z}$ and the electric displacement D^φ vanish identically. The other equations are written down in polar coordinates directly. The strain-displacement relations are:

$$S_{rr} = \frac{\partial U}{\partial r}, \quad S_{\varphi\varphi} = rU,$$
$$S_{zz} = \frac{\partial W}{\partial z}, \quad S_{rz} = \tfrac{1}{2}\left(\frac{\partial U}{\partial z} + \frac{\partial W}{\partial r}\right). \tag{4}$$

The equation of motion in the r-direction is:

$$\frac{\Delta}{\Delta r} T^{rr} - rT^{\varphi\varphi} + \frac{\partial}{\partial z} T^{rz} + \rho\omega^2 U = 0, \tag{5}$$

and in the z-direction:

$$\frac{\Delta}{\Delta r} T^{rz} + \frac{\partial}{\partial z} T^{zz} + \rho\omega^2 W = 0, \tag{6}$$

where ρ denotes the mass-density and:

$$\frac{\Delta}{\Delta r} = \frac{\partial}{\partial r} + \frac{1}{r}. \tag{7}$$

By virtue of the axisymmetry the equation of motion in the φ-direction is satisfied identically. Finally the quasi-static Maxwell equations are:

$$E_r = -\frac{\partial V}{\partial r}, \quad E_z = -\frac{\partial V}{\partial z}, \tag{8}$$

$$\frac{\Delta}{\Delta r} D^r + \frac{\partial}{\partial z} D^z = 0. \tag{9}$$

When we substitute (4) and (8) into (2) and (3), and then the results into (5), (6) and (9), we obtain three coupled partial differential equations for U, W and V, valid in $|z| < 1$:

$$\begin{pmatrix} c^{1111}\frac{\partial}{\partial r}\frac{\Delta}{\Delta r} + \rho\omega^2 + c^{1313}\frac{\partial^2}{\partial z^2} & (c^{1133} + c^{1313})\frac{\partial^2}{\partial r\partial z} & (e^{113} + e^{311})\frac{\partial^2}{\partial r\partial z} \\ (c^{1133} + c^{1313})\frac{\Delta}{\Delta r}\frac{\partial}{\partial z} & c^{1313}\frac{\Delta}{\Delta r}\frac{\partial}{\partial r} + \rho\omega^2 + c^{3333}\frac{\partial^2}{\partial z^2} & e^{113}\frac{\Delta}{\Delta r}\frac{\partial}{\partial r} + e^{333}\frac{\partial^2}{\partial z^2} \\ (e^{113} + e^{311})\frac{\Delta}{\Delta r}\frac{\partial}{\partial z} & e^{113}\frac{\Delta}{\Delta r}\frac{\partial}{\partial r} + e^{333}\frac{\partial^2}{\partial z^2} & -\epsilon^{11}\frac{\Delta}{\Delta r}\frac{\partial}{\partial r} - \epsilon^{33}\frac{\partial^2}{\partial z^2} \end{pmatrix} \begin{pmatrix} U \\ W \\ V \end{pmatrix} = 0.$$

$$|z| < 1 \tag{10}$$

In the region $|z| > 1$ we first give equations valid for the more general case, that the plate is surrounded by a medium with dielectric constant ϵ^0. The equations are (8) and (9) and:

$$D^r = \epsilon^0 E_r, \quad D^z = \epsilon^0 E_z, \tag{11}$$

Substitution of (8) into (11) and the result into (9) gives:

$$\left(\frac{\Delta}{\Delta r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}\right) V = 0. \qquad |z| > 1. \tag{12}$$

In the case that the plate is surrounded by vacuum, the value of the ϵ^0 in (11) equals the value of the dielectric constant in vacuum, which will be denoted by ϵ^v.

The surfaces at $z = \pm 1$ are traction free, which yields two equations:

$$T^{zz} = T^{rz} = 0 \quad z = \pm 1, \quad 0 \leqq r < \infty. \tag{13}$$

Finally, the jump in the normal component of the electric displacement equals the free-charge density at the surface, which is denoted by q_+ for the upper surface and by q_- for the lower one:

$$D^z(z = \pm 1 + 0) - D^z(z = \pm 1 - 0) = q_\pm \quad 0 \leqq r < \infty. \tag{14}$$

This must be considered as two equations: one with the upper level and one with the lower level of the ±signs. The eqns (13) and (14) can again be expressed in U, W and V, when we use the eqns (4), (8), (2), (3) and (11):

$$\left.\begin{pmatrix} c^{1313}\dfrac{\partial}{\partial z} & c^{1313}\dfrac{\partial}{\partial r} & e^{113}\dfrac{\partial}{\partial r} \\ c^{1133}\dfrac{\Delta}{\Delta r} & c^{3333}\dfrac{\partial}{\partial z} & e^{333}\dfrac{\partial}{\partial z} \\ e^{311}\dfrac{\Delta}{\Delta r} & e^{333}\dfrac{\partial}{\partial z} & -\epsilon^{33}\dfrac{\partial}{\partial z} \end{pmatrix}\begin{pmatrix} U \\ W \\ V \end{pmatrix}\right|_{z=\pm 1\mp 0} + \left.\begin{pmatrix} 0 \\ 0 \\ \epsilon^0\dfrac{\partial V}{\partial z} \end{pmatrix}\right|_{z=\pm 1\pm 0} = \begin{pmatrix} 0 \\ 0 \\ \mp q_\mp \end{pmatrix}. \tag{15}$$

The free-charge density is non-zero only where there are electrodes present, hence:

$$q_+(r) = q_-(r) = 0, \qquad r \notin C_e \tag{16}$$

where:

$$C_e = [0, a_1] \cup [a_2, a_3]. \tag{17}$$

The electric potential is constant on each electrode, hence:

$$\left.\begin{matrix} V_{(r,z=1)} = V_1 \\ V_{(r,z=-1)} = V_3 \end{matrix}\right\} 0 \leqq r \leqq a_1; \qquad \left.\begin{matrix} V_{(r,z=1)} = V_2 \\ V_{(r,z=-1)} = V_4 \end{matrix}\right\} a_2 \leqq r \leqq a_3 \tag{18}$$

where V_1, V_2, V_3 and V_4 are the potentials on the electrodes.

When the values of V_1, V_2, V_3 and V_4 are prescribed, then the U, W, V and $q_\pm$ follow from (10), (12), (15), (16) and (18). Note that U and W are defined in $|z| \leqq 1$, $0 \leqq r < \infty$ and that V is defined in $-\infty \leqq z \leqq \infty$, $0 \leqq r \leqq \infty$. They denote the complex amplitude of the corresponding physical quantities.

3. DECOMPOSITION OF THE ADMITTANCE MATRIX

In this section we will decompose a solution of the eqns (10), (12), (15), (16) and (18) into two parts, being its symmetric part and its antisymmetric part. Using symmetry-properties of these parts we will give an expression for the admittance-matrix of the plate with electrodes.

Let the electric potentials V_1, V_2, V_3 and V_4 on the electrodes be prescribed with $V_1 = V_3$ and $V_2 = V_4$. Then it can be shown (see [4]) for the solution of the eqns (10), (12), (15), (16) and (18) induced by these potentials, that the W and V are symmetric functions of z, that the U is an antisymmetric function of z and that $q_+(r) = q_-(r)$. We call the wave represented by this solution an anti-symmetric wave. The total free charges Q_2, Q_2, Q_3 and Q_4 on the electrodes satisfy $Q_1 = Q_3$ and $Q_2 = Q_4$ in the case of an anti-symmetric wave. Moreover, they are linear expressions in V_1 and V_2:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} Q_3 \\ Q_4 \end{pmatrix} = \begin{pmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{12} & \sigma_{22} \end{pmatrix}\begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \tag{19}$$

where the matrix σ_{ij} is symmetric, as can be derived from more general reciprocity relations for piezoelectric material. The σ_{ij} depend on the frequency ω, the material constants and the geometric dimensions of the plate with electrodes.

Let the potentials on the electrodes be prescribed with $V_1 = -V_3$ and $V_2 = -V_4$. Then it can be shown for the solution induced by these potentials, that W and V are anti-symmetric in z, that U is symmetric in z and that $q_+(r) = -q_-(r)$. We call the wave represented by this solution a symmetric wave. The total free charges on the electrodes satisfy $Q_1 = -Q_3$ and $Q_2 = -Q_4$ in the case of a symmetric wave, and:

$$\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} -Q_3 \\ -Q_4 \end{pmatrix} = \begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{12} & \alpha_{22} \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \end{pmatrix}, \tag{20}$$

where the α_{ij} depend again on the frequency, the material constants and the geometric dimensions.

Finally, let the potentials on the electrodes be arbitrary. Then we write:

$$\begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix} = V_1^s \begin{pmatrix} 1 \\ 0 \\ 1 \\ 0 \end{pmatrix} + V_2^s \begin{pmatrix} 0 \\ 1 \\ 0 \\ 1 \end{pmatrix} + V_1^a \begin{pmatrix} 1 \\ 0 \\ -1 \\ 0 \end{pmatrix} + V_2^a \begin{pmatrix} 0 \\ 1 \\ 0 \\ -1 \end{pmatrix}, \tag{21}$$

where V_1^s, V_2^s, V_1^a and V_2^a are coefficients. The wave induced by only the first two terms on the right-hand-side of (21) is anti-symmetric and the wave induced by the last two terms is symmetric. By linearity, the sum of these two waves is the wave induced by V_1, V_2, V_3 and V_4. Moreover, using (19)–(21) we derive that:

$$\begin{pmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} \sigma_{11} & \sigma_{12} & \alpha_{11} & \alpha_{12} \\ \sigma_{12} & \sigma_{22} & \alpha_{12} & \alpha_{22} \\ \sigma_{11} & \sigma_{12} & -\alpha_{11} & -\alpha_{12} \\ \sigma_{12} & \sigma_{22} & -\alpha_{12} & -\alpha_{22} \end{pmatrix} \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{pmatrix} \begin{pmatrix} V_1 \\ V_2 \\ V_3 \\ V_4 \end{pmatrix}. \tag{22}$$

It follows from (22), that the admittance matrix of the plate with electrodes is determined by the σ_{ij} and the α_{ij}.

In this paper we will discuss anti-symmetric waves only. The W and V are symmetric in z and U is anti-symmetric in z, and hence it is sufficient to consider the range $z \geqq 0$ only. Note that when the potentials are prescribed symmetrically on the electrodes, then they are in fact prescribed potential-differences with respect to the potential at infinity, which is assumed to be zero. Since $q_+ = q_-$, we omit the subscripts + and −.

4. A HEURISTIC DISCUSSION

The value of the dielectric constants ϵ^{11} and ϵ^{33} of the plate considered in this paper is large with respect to the dielectric constant of vacuum. Therefore it gives a good notion of the behaviour of the plate surrounded by vacuum, when we do not give results corresponding to the case that the ϵ^0, introduced in (11), equals the dielectric constant of vacuum ϵ^v only, but also give results corresponding to $\epsilon^0 > \epsilon^v$ and $\epsilon^v > \epsilon^0 \geqq 0$. The case that $\epsilon^0 > \epsilon^v$ corresponds to the plate surrounded by some dielectric medium and the case that $\epsilon^v > \epsilon^0 \geqq 0$ must be considered as a "mathematical idealization".

We make two statements concerning the limit $\epsilon^0 \downarrow 0$ of the waves induced by symmetrically prescribed potentials on the electrodes. The consequences of these will be checked numerically in the next section. Here we give a heuristic discussion of them. (i) For the normal component of the electric displacement at $z = 1+0$ we have:

$$\lim_{\epsilon^0 \downarrow 0} D^z_{(r, z=1+0)} = 0 \quad 0 \leqq r < \infty. \tag{23}$$

When we assume for this vibration that in the limit $\epsilon^0 \downarrow 0$, the electric potential and its partial derivatives tend to a finite limit, then (23) follows from (11) and (8). (ii) For the total free charge at the upper electrodes, Q, we have:

$$\lim_{\epsilon^0 \downarrow 0} Q = \lim_{\epsilon^0 \downarrow 0} 2\pi \int_0^\infty rq(r)\,dr = 0. \tag{24}$$

Here the integral over r is in fact over C_e (see 17), since q vanishes outside C_e. In order to make (24) plausible we derive a result concerning the normal component of the electric displacement at $z = 1-0$. The eqn (9) is multiplied by r and integrated over $|z|<1$ and $0<r<R$ (R arbitrary). Applying Gauss' theorem and using the symmetry-properties we obtain:

$$\int_0^R D^z(r, z=1-0)r\,dr + R\int_0^1 D^r(r=R, z)\,dz = 0. \tag{25}$$

We assume a "small" damping to be present in the material. Then an asymptotic expression for D^r can be derived (see [4]):

$$D^r = \frac{1}{r^2}\frac{Q}{2\pi\epsilon^0}\left\{\epsilon^{11} + \frac{(e^{113})^2}{c^{1313}}\frac{\cos\lambda z}{\cos\lambda}\right\} + 0(r^{-3}),\ r\to\infty|z|<1 \tag{26}$$

where:

$$\lambda = \omega\left(\frac{\rho}{c^{1313}}\right)^{1/2}. \tag{27}$$

Hence if $\cos\lambda \neq 0$, then by (25) and (26):

$$\int_0^\infty D^z(r, z=1-0)r\,dr = 0. \tag{28}$$

The free charge density equals the jump in the normal component of the electric displacement, by virtue of (14). It follows from (28), that the contribution of $D^z(z=1-0)$ to the total free charge vanishes, and hence:

$$Q = 2\pi\int_0^\infty D^z(r, z=1+0)r\,dr. \tag{29}$$

Now the eqn (24) follows from (29) and (23), when it can be shown that the change of order of taking the limit $\epsilon^0 \to 0$ and integration with respect to r is allowed. Setting $D^z(z=1+0)=0$ is a common practice in many papers treating piezoelectric problems, however in our case its validity needs a proof. Therefore this has to be considered indeed as a heuristic discussion, which will be checked numerically in the next section.

5. THE LIMIT $\epsilon^0 \to 0$

In this section we give numerical results for the values of the σ_{ij}, introduced in (19), for different values of ϵ^0. For the numerical method, we refer to the Appendix.

It will appear that the σ_{ij} are complex in general, which implies that there is energy-dissipation by the plate. In the computations we have neglected dissipation in the material (the c^{ijkl}, e^{klj} and ϵ^{ik} are real) and dissipation of energy is completely due to radiation of energy is completely due to radiation to $r=\infty$.

First we express the σ_{ij} in three quantities σ_1, σ_2 and σ_3. Equation (19) is equivalent to:

$$\begin{pmatrix} V_1 \\ V_2 \end{pmatrix} = \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} \tag{30}$$

where:

$$\begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{pmatrix}=\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}^{-1}. \tag{31}$$

We introduce:

$$\begin{aligned}\sigma_1&=(\tau_{11}+\tau_{22}-2\tau_{12})^{-1},\\ \sigma_2&=(\tau_{11}-\tau_{22})^{-1},\\ \sigma_3&=(\tau_{11}+\tau_{22})^{-1}.\end{aligned} \tag{32}$$

Then the τ_{ij} are expressed in the σ_k by:

$$\begin{pmatrix}\tau_{11} & \tau_{12}\\ \tau_{12} & \tau_{22}\end{pmatrix}=\frac{1}{2\sigma_1}\begin{pmatrix}0 & -1\\ -1 & 0\end{pmatrix}+\frac{1}{2\sigma_2}\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}+\frac{1}{2\sigma_3}\begin{pmatrix}1 & 1\\ 1 & 1\end{pmatrix}, \tag{33}$$

and the σ_{ij} are expressed in the σ_k by:

$$\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}=\sigma_1\left[\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}+\sigma_3\begin{pmatrix}-\frac{1}{\sigma_2} & \frac{1}{\sigma_1}\\ \frac{1}{\sigma_1} & \frac{1}{\sigma_2}\end{pmatrix}\right]\times\left[1-\frac{\sigma_3}{2}\left(\frac{1}{\sigma_1}+\frac{\sigma_1}{\sigma_2^2}\right)\right]^{-1} \tag{34}$$

Equation (30) gives the electric potentials on the electrodes, when the total free charges on the electrodes are prescribed (symmetrically, i.e. $Q_3=Q_1$ and $Q_4=Q_2$). When we choose the free charges such that $Q=Q_1+Q_2$ vanishes, then $Q_2=-Q_1$ and by (30) and (32):

$$\sigma_1(V_1-V_2)=Q_1;\quad \sigma_2(V_1+V_2)=Q_1 \tag{35}$$

It follows from (35) that the σ_1 and σ_2 completely determine the electric potentials on the electrodes which are due to prescribed free charges for which the total free charge on the upper electrodes, Q, vanishes. It will appear below that σ_1 and σ_2 are almost independent of ϵ^0 for $0\leqq\epsilon^0\leqq\epsilon^v$, while:

$$\lim_{\epsilon^0\downarrow 0}\sigma_3=0. \tag{36}$$

The σ_{ij} have been computed for the piezoelectric ceramic (see [7]):

$$\left.\begin{aligned}c^{1111}&=8.98\\ c^{1133}&=3.88\\ c^{1313}&=2.60\\ c^{3333}&=8.23\end{aligned}\right\}10^{10}\,\mathrm{N\,m^{-2}}\qquad \left.\begin{aligned}e^{113}&=13.4\\ e^{311}&=-9.24\\ e^{333}&=15.5\end{aligned}\right\}\mathrm{NV^{-1}\,m^{-1}}$$

$$\rho=7.5\quad 10^3\,\mathrm{kg\,m^{-3}}\qquad \left.\begin{aligned}\epsilon^{11}&=9.02\\ \epsilon^{33}&=6.69\end{aligned}\right\}10^{-9}\,\mathrm{Vm^{-1}}. \tag{37}$$

The numerical results in this paper are given for a plate with thickness 1 mm; i.e. $h=0.5\;10^{-3}$ m, while in the formulae we assume $h=1$. The complex values of the σ_k are plotted in Fig. 2 as functions of $^{10}\log(\epsilon^0/\epsilon^v)$ for two values of the frequency ω. It appears from these figures that the σ_k depend strongly on ϵ^0 when ϵ^0 is of the order of ϵ^{11} or ϵ^{33}. In $0<\epsilon^0\leqq\epsilon^v$ the σ_1 and σ_2 are indeed almost constant and σ_3 decreases slowly to zero when $\epsilon^0\to 0$. From these numerical results and (34) it follows that:

$$\lim_{\epsilon^0\downarrow 0}\begin{pmatrix}\sigma_{11} & \sigma_{12}\\ \sigma_{12} & \sigma_{22}\end{pmatrix}=\sigma_1\begin{pmatrix}1 & -1\\ -1 & 1\end{pmatrix}. \tag{38}$$

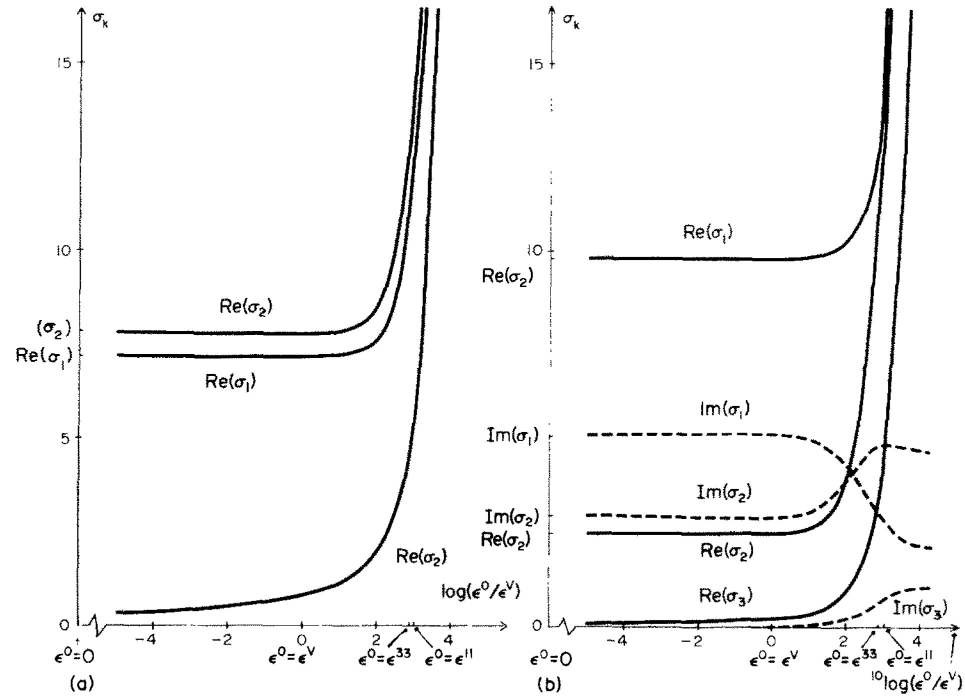

Fig. 2. The σ_1, σ_2 and σ_3, introduced in (32), as functions of $^{10}\log(\epsilon^0/\epsilon^v)$. Scale-unit is $h\epsilon^{11}$. Material constants (37) and $a_1/h = 1$, $a_2/h = 2$, $a_3/h = 3$. In (a) $\omega = 3.0\,10^6$ rad. sec^{-1} and in (b) $\omega = 8.0\,10^6$. The imaginary parts of the σ_k in (a) have not been plotted, since they are less than 0.1 $h\epsilon^{11}$ The values given on the left-hand vertical axis ($\epsilon^0 = 0$) correspond to the numerical method described in Section 6.

When the right-hand-side of (38) is substituted in (19), then the total free charge on the upper electrodes, Q, vanishes indeed for arbitrary values of V_1 and V_2 for $\epsilon^0 \to 0$, as has been made plausible at the end of Section 4.

Note that the matrix at the right-hand-side of (38) is singular. However, the matrix in (19) is regular for $\epsilon^0 > 0$ and using (33) and (36) we obtain for its inverse

$$\lim_{\epsilon^0 \downarrow 0} \sigma_3 \begin{pmatrix} \tau_{11} & \tau_{12} \\ \tau_{12} & \tau_{22} \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}. \tag{39}$$

6. THE CASE $\epsilon^0 = 0$

In this section we assume that

$$D^z(r, z = 1 + 0) = 0 \quad 0 \leqq r < \infty \tag{40}$$

according to the common practice mentioned above. Then we will show that the values of σ_1 and σ_2 corresponding to $\epsilon^0 = 0$ can be computed.

We will determine a symmetric solution of the eqns (10) in $|z| < 1$, which satisfies the following conditions at $z = 1$: (i) the mechanical conditions (13), (ii) the electrical conditions (14) with (40), (iii) the condition (16) for the free charge density, and besides:

$$Q = 2\pi \int_{C_e} q(r) r \, \mathrm{d}r = 0 \tag{41}$$

(iv) the electric potential is constant in $0 \leqq r \leqq a_1$ and in $a_2 \leqq r \leqq a_3$. We disregard the electric potential in $|z| > 1$. Since Q is zero, we obtain the value of σ_1 and σ_2 for $\epsilon^0 = 0$ from this solution by:

$$\sigma_1 = \frac{Q_1}{V_1 - V_2}, \quad \sigma_2 = \frac{Q_1}{V_1 + V_2}, \tag{42}$$

as follows from (35).

By virtue of the axisymmetry we use the Hankel transformation. If $f = f(r, z)$, then $\bar{f}^{\nu}(\xi, z)$ is defined by:

$$\bar{f}^{\nu}(\xi, z) = \int_0^{\infty} rf(r, z)J_{\nu}(\xi r)\,dr \tag{43}$$

where J_{ν} denotes the Bessel function of the first kind of order ν. The inverse transformation is the same transformation:

$$f(r, z) = \int_0^{\infty} \xi \bar{f}^{\nu}(\xi, z)J_{\nu}(\xi r)\,d\,\xi. \tag{44}$$

For a symmetric solution of (10), which satisfies (i) and (ii) at $z = 1$, it can be shown that (see Appendix eqn (A5)):

$$\bar{V}^0(\xi, z = 1) = K(\xi)\bar{q}^0(\xi), \quad 0 \leqq \xi < \infty \tag{45}$$

where the value of $K(\xi)$ depends on ξ, ω and the material constants. If ω is fixed, then K is a meromorphic function of ξ and:

$$\begin{aligned} K(\xi) &= \text{const.}\,\xi^{-2} + O(1), \qquad \xi \to 0 \\ K(\xi) &= \text{const.}\,\xi^{-1} + O(\xi^{-3}) \quad 0 < \xi \to \infty. \end{aligned} \tag{46}$$

When we apply the inverse Hankel transformation (44) to (45), we obtain an integral expression for $V(r, z = 1)$. We investigate the behaviour at $\xi = 0$ of the integrand in this expression. Since $\bar{q}^0$ is symmetric and analytic in ξ, it is $O(\xi^2)$ at $\xi = 0$, provided that

$$\bar{q}^0(0) = \int_{C_e} q(r)r\,dr = \frac{1}{2\pi}Q \tag{47}$$

vanishes. Hence by virtue of (41), the order of the integrand under consideration is $O(\xi)$ at $\xi = 0$.

We must restrict ourselves to free-charge densities which satisfy (41). However, it will appear to be convenient to consider arbitrary free-charge densities, which vanish identically outside the electrodes, and to subtract a free-charge density, which equals a constant value on the electrodes, such that the resulting free-charge density $\tilde{q}$ satisfies (41). When q is arbitrary with $q(r) = 0$ for $r \notin C_e$, then $\tilde{q}$ is defined by:

$$\tilde{q}(r) = q(r) - \alpha \int_{C_e} q(s)s\,ds.\,\theta(r), \quad 0 \leqq r < \infty \tag{48}$$

where the function θ is:

$$\theta(r) = 1 \quad r \in C_e; \quad \theta(r) = 0 \quad r \notin C_e \tag{49}$$

and the number α:

$$\alpha^{-1} = \int_{C_e} r\,dr = \frac{1}{2}(a_3^2 - a_2^2 + a_1^2) \tag{50}$$

We compute the electric potential at $z = 1$ induced by $\tilde{q}$, using (45) and the inverse Hankel transformation:

$$\begin{aligned} V(r, z = 1) &= \int_0^{\infty} \xi K(\xi)\bar{\tilde{q}}^0(\xi)J_0(\xi r)\,d\xi \\ &= \int_0^{\infty} \xi K(\xi)J_0(\xi r)\,d\xi \int_{C_e} s\tilde{q}(s)J_0(\xi s)\,ds. \end{aligned} \tag{51}$$

A change of the order of integrations gives after substitution of (48):

$$\int_{C_e} s\tilde{q}(s)J_0(\xi s)\,ds = \int_{C_e} sq(s)\,ds\left\{J_0(\xi s) - \alpha\int_{C_e} tJ_0(\xi t)\,dt\right\}. \tag{52}$$

Substitution of (52) in (51) and a change of order of the integration with respect to ξ and s gives:

$$V(r, x = 1) = \int_{C_e} sL(r, s)q(s)\,ds, \tag{53}$$

where:

$$L(r, s) = \int_0^\infty \xi K(\xi)J_0(\xi r)\left\{J_0(\xi s) - \alpha\int_{C_e} tJ_0(\xi t)\,dt\right\}d\xi. \tag{54}$$

The $L(r, s)$ has a logarithmic singularity at $r = s$, as can be derived using the expansion (46) for $K(\xi)$ for $\xi \to \infty$. There are poles of $K(\xi)$ on the path of integration. They are integrated in the same way as described in the Appendix.

The electric potential is constant on the electrodes, V_1 on electrode 1 and V_2 on electrode 2. Hence (53) can be considered as an integral equation of the first kind for q:

$$\int_{C_e} sL(r, s)q(s)\,ds = V_1, \quad 0 \leqq r \leqq a_1$$

$$= V_2, \quad a_2 \leqq r \leqq a_3. \tag{55}$$

If $q = \Theta$, then $\tilde{q}$ vanishes identically, which implies that $V(r, z = 1)$ given by (51) also vanishes identically. Hence if $V_1 = V_2 = 0$, then $q = \Theta$ is a solution of (55). Therefore, if a solution q of (55) exists, then, by linearity, $q + \mu \cdot \Theta$ is also a solution for arbitrary values of the number μ. However, the $\tilde{q}$ corresponding to this class of solutions does not depend on μ.

Since the solution q of (55) is not unique, we expect from Fredholm operator theory that V_1 and V_2 cannot be chosen arbitrarily. We prescribe V_1 and consider V_2 to be determined such that a solution q of (55) exists. Then V_1, V_2 and $\tilde{q}$ are the potentials and free-charge densities on the electrodes belonging to the solution of (10) in $|z| < 1$, which satisfies at $z = 1$ the conditions (i), . . . ,(iv) mentioned at the beginning of this section. We now describe the numerical solution of the eqn (55).

We choose $n + m$ collocation points r_k on C_e:

$$0 < r_1 < r_2 \ldots . < r_n = a_1 < a_2 = r_{n+1} < \ldots . < r_{n+m} = a_3 \tag{56}$$

and we introduce linear interpolation functions $f_k(s)$, which have the value one or zero at the collocation points (see Fig. 3). The function f_1 is chosen in a special way, due to the axisymmetry. We approximate the solution q of (55) by an expression of the form:

$$q(s) = \sum_{k=1}^{n+m} q_k f_k(s)R(s), \quad s \in C_e \tag{57}$$

where:

$$R(s) = \{|a_1 - s| \cdot |a_2 - s| \cdot |a_3 - s|\}^{-1/2}. \tag{58}$$

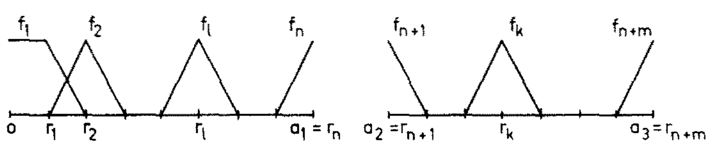

Fig. 3. The coordinate-functions $f_k(s)$.

The q_k are coefficients and the function $R(s)$ is introduced, since we expect, by virtue of the logarithmic singularity of L at $r = s$, the free-charge density to have a square-root singularity at the edge of the electrodes. Collocation at the points r_k gives $n+m$ equations for q_k:

$$\begin{aligned}\sum_{k=1}^{n+m} M_{lk}q_k &= V_1, \qquad 1 = 1,\ldots,n \\ &= V_2, \qquad 1 = n+1,\ldots,n+m\end{aligned} \tag{59}$$

where:

$$M_{lk} = \int_{C_e} sL(r_l, s)f_k(s)R(s)\,\mathrm{d}s. \tag{60}$$

Since $q = \Theta$ satisfies (55) with $V_1 = V_2 = 0$, the matrix (M_{k1}) in (59) is singular, if we disregard the error introduced by the discretization. We will prescribe $V_1 = 1$ and determine V_2 such that a solution of (59) exists. We do this as follows:

Discretization of (41) yields for the q_k:

$$\sum_{k=1}^{n+m} T_k q_k = 0 \tag{61}$$

where:

$$T_k = \int_{C_e} sf_k(s)R(s)\,\mathrm{d}s. \tag{62}$$

The eqns (59) and (61) give $n+m+1$ equations for the unknowns $q_1,\ldots,q_{n+m}$ and V_2:

$$\begin{bmatrix} M_{1,1}\cdots\cdots M_{1,n+m} & 0 \\ \vdots & \vdots \\ M_{n,1}\cdots\cdots M_{n,n+m} & 0 \\ M_{n+1,1}\cdots\cdots M_{n+1,n+m} & -1 \\ \vdots & \vdots \\ M_{n+m,1}\cdots\cdots M_{n+m,n+m} & -1 \\ T_1\cdots\cdots T_{n+m} & 0 \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ \vdots \\ \vdots \\ \vdots \\ q_{n+m} \\ V_2 \end{bmatrix} = \begin{bmatrix} V_1 \\ \vdots \\ V_1 \\ 0 \\ \vdots \\ 0 \\ 0 \end{bmatrix}. \tag{63}$$

The matrix in (63) appeared to be regular during the numerical computations. When q_k and V_2 are determined by (63), then q, given by (57), satisfies (55) and (41). The total free charge on electrode 1, Q_1, follows from a numerical integration, and the values of σ_1 and σ_2 follows from (42).

Numerical values of σ_1 and σ_2, obtained in this way, are given on the left-hand vertical axes of the Figs. 2a and b. They are in good agreement with the values of σ_1 and σ_2 for values of ϵ^0 with $0 < \epsilon^0 \leqq \epsilon^v$. This agreement holds for all values of ω, except for some values, where certain resonances occur. This implies that the common practice to set the normal component of the electric displacement at $z = 1+0$ equal to zero is justified, except for the mentioned resonances. These resonances will be investigated in the next two sections.

7. DISPERSION CURVES

In order to discuss the resonances we first introduce the dispersion curves. These curves correspond to a plate which is either without electrodes or completely covered on both sides with electrodes. We will investigate the curves for $\epsilon^0 = \epsilon^v$ and for $\epsilon^0 = 0$.

We begin with the case that the plate is without electrodes and take $\epsilon^0 = \epsilon^v$. Then the right-hand-side of (15) vanishes identically. We consider expressions of the form:

$$\begin{pmatrix} U \\ W \\ V \end{pmatrix}_{(r,z)} = \begin{pmatrix} J_1(\xi r)\tilde{U}_{(z)} \\ J_0(\xi r)\tilde{W}_{(z)} \\ J_0(\xi r)\tilde{V}_{(z)} \end{pmatrix} \tag{64}$$

where J_0 and J_1 are Bessel functions of the first kind. Substitution of (64) into (10), (12) and (15) (with vanishing right-hand-side) gives ordinary differential equations for $\tilde{U}$, $\tilde{W}$ and $\tilde{V}$. The dispersion curves consist of those pairs of values of ξ and ω for which this system of ordinary differential equations for $\tilde{U}$, $\tilde{W}$ and $\tilde{V}$ has a solution, which tends to zero for $|z|\rightarrow\infty$. It is sufficient for our aim to consider positive real values of ξ and ω only, as is explained in the Appendix. Moreover, we consider anti-symmetric waves only, which implies that $\tilde{U}$ in (64) must be anti-symmetric in z and that $\tilde{W}$ and $\tilde{V}$ must be symmetric in z.

The dispersion curves have been computed for the material constants (37). In the case that $\epsilon^0 = \epsilon^v$ they are denoted by $\xi_1{}^s$, $\xi_2{}^s, \ldots$ and are plotted with full lines in Fig. 4. The lowest frequency corresponding to a fixed dispersion curve is called the cut-off frequency. These frequencies are denoted by $\omega_1{}^s$, $\omega_2{}^s, \ldots$.

In the case that $\epsilon^0 = 0$ we disregard the electric potential outside the plate and the dispersion curves for a plate without electrodes are introduced as consisting of those pairs of values of ξ and ω at which the expression (64) can represent an anti-symmetric solution of the eqns (10) and (15) (with $\epsilon^0 = 0$ and with vanishing right-hand-side). These curves are denoted by $\tilde{\xi}_1{}^s$, $\tilde{\xi}_2{}^s, \ldots$ and are plotted with full lines also in Fig. 4. The corresponding cut-off frequencies are denoted by $\tilde{\omega}_1{}^s$, $\tilde{\omega}_2{}^s, \ldots$

Next we consider a plate with infinite electrodes. Moreover the electric potential must vanish on the electrodes:

$$V(r, z = \pm 1) = 0 \qquad 0 \leqq r < \infty. \tag{65}$$

The dispersion curves for a plate with infinite electrodes are the same for all values of $\epsilon^0 \geqq 0$. They consist of those pairs of values of ξ and ω for which the expression (64) can represent a symmetric solution of (10), of the first two equations in (15), and of (65). They are denoted by $\xi_1{}^\sigma$, $\xi_2{}^\sigma, \ldots$, and are plotted with dotted lines in Fig. 4. The $\xi_1{}^\sigma$-curve has not been plotted seperately,

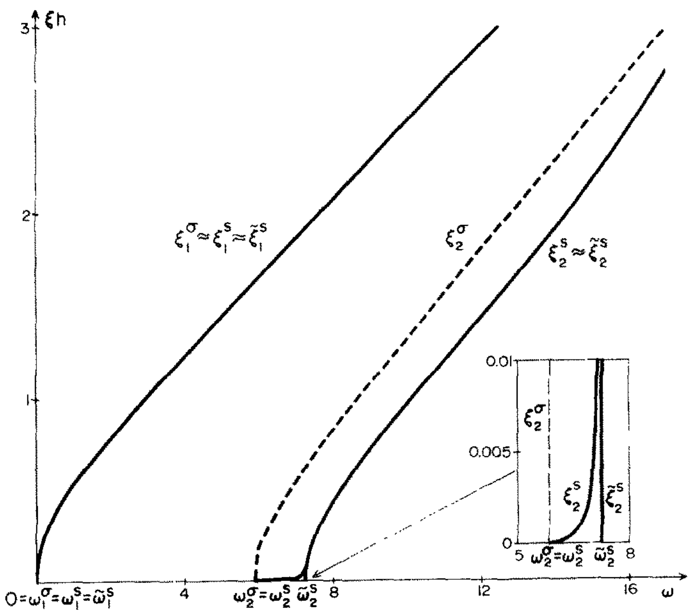

Fig. 4. Dispersion curves for anti-symmetric waves in a plate without electrodes. (——) and with infinite electrodes (-----). The curves for a plate without electrodes depend on ϵ^0: $\xi_k{}^s$ for $\epsilon^0 = \epsilon^v$ and $\tilde{\xi}_k{}^s$ for $\epsilon^0 = 0$. Scale-unit for ω is 10^6 rad. sec^{-1}. The scale for ξh is stretched a factor 100 in the enlarged detail. Material constants (37).

since it almost coincides with the ξ_1^s-curve. The corresponding cut-off frequencies are denoted by ω_1^σ, ω_2^σ, ...

Inspection of Fig. 4 shows that all dispersion curves emanate from the line $\xi = 0$. Frequencies at which dispersion curves emanate from this line satisfy:

$$\sin \mu = 0 \tag{66}$$

or:

$$\cos \lambda = 0 \tag{67}$$

or:

$$(e^{113})^2 tg\lambda + c^{1313}\epsilon^{11}\lambda = 0, \tag{68}$$

where λ is given in (27) and μ is given by:

$$\mu = \omega\rho^{1/2}(c^{3333} + (e^{333})^2/\epsilon^{33})^{-1/2} \tag{69}$$

At $\omega = 0$ eqn (66) is satisfied and there three dispersion curves emanate from the line $\xi = 0$: ξ_1^σ, ξ_1^s and $\tilde{\xi}_1^s$. These curves almost coincide and only one of them is plotted. Frequencies corresponding to $\mu = k \cdot \pi$, $k = 1,2,\ldots$ are beyond the frequency range considered in this paper.

The frequency $\omega_2^\sigma = \omega_2^s$ corresponds to $\lambda = \frac{1}{2}\pi$. At this frequency the curves ξ_2^σ and ξ_2^s emanate from the line $\xi = 0$. The ξ_2^σ-curve meets the line $\xi = 0$ perpendicularly. For frequencies which are slightly greater than ω_2^σ an approximate expression for ξ_2^s can be derived (see [4]):

$$\xi_2^s \approx -\epsilon^0\left(\epsilon^{11} + \frac{(e^{113})^2}{c^{1313}}\frac{tg\lambda}{\lambda}\right)^{-1}. \tag{70}$$

Hence if ϵ^0 is small, then the ξ_2^s-curve makes a small angle with the line $\xi = 0$. It appears from Fig. 4 that this angle is indeed small for the material constants (37) and $\epsilon^0 = \epsilon^v$.

The frequency $\tilde{\omega}_2^s$ corresponds to the lowest positive solution of (68). There the $\tilde{\xi}_2^s$-curve emanates at an angle of 90 degrees from the line $\xi = 0$. For not too small values of ξ this curve almost coincides with the ξ_2^s-curves.

8. RESONANCES

In this section we discuss the resonances of the unbounded plate with two pairs of electrodes surrounded by vacuum (see Fig. 1). As appears from the numerical results given in the Figs. 5–7, the values of the σ_{ij} represent peaks when they are plotted as functions of ω. These peaks are called the resonances of the plate with electrodes. In this paper we restrict ourselves to the lowest resonant-frequencies: they are in the interval $[\omega_2^\sigma, \tilde{\omega}_2^s]$. At the numerical computations we have taken into account the electric field in the vacuum outside the plate. However, we will also discuss what results would have been obtained if we had applied the common practice mentioned in Section 1.

We remark that the frequency ω_2^σ corresponds to $\lambda = \pi/2$, which satisfies (67). At this frequency an expression of the form:

$$\begin{pmatrix} U \\ W \\ V \end{pmatrix}_{(r,z)} = \begin{pmatrix} r.\sin \pi/2z \\ \tilde{W}(z) \\ \tilde{V}(z) \end{pmatrix} \tag{71}$$

represents a solution of the eqn (10), of the first two eqns in (15), and of (65). Here $\tilde{W}_{(z)}$ and $\tilde{V}_{(z)}$ are symmetric functions of z. The expression (71) represents an anti-symmetric thickness vibration of the plate with infinite electrodes. Note that the ratio between the amplitudes of U and of W depends on r: for r small the vibration is a flexural vibration; for r large the vibration resembles a thickness-shear vibration (see [3]). The numerical results in this paper concern the

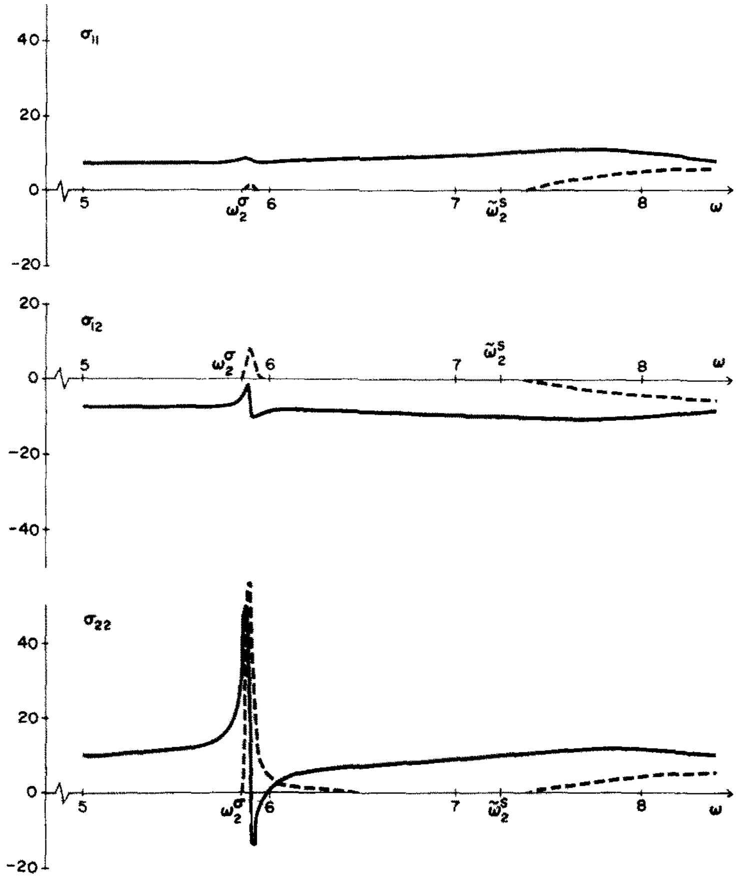

Fig. 5. The σ_{ij}, as functions of ω. Scale-unit is $h\epsilon^{11}$; real part (——); imaginary part, (----). Scale-unit for ω is 10^6 rad. sec^{-1}. Radii of the electrodes: $a_1/h = 1$; $a_2/h = 2$; $a_3/h = 3$. The ϵ^0 equals the dielectric constant of vacuum. For dispersion curves and material constants see Fig. 4.

values of the σ_{ij} only. However, it is likely that at the resonant-frequencies in the interval $[\omega_2^\sigma, \tilde{\omega}_2^s]$, the vibration of the plate with bounded electrodes is, for r not too large, approximately given by (71).

As appears from numerical results not plotted here, the resonances in the interval $[\omega_2^\sigma, \tilde{\omega}_2^s]$ can be devided into two types. At one type the value of σ_1 has a peak, while σ_2 and σ_3 are smooth when they are considered as functions of ω in the neighbourhood of the resonant-frequency. At the resonant-frequency of the other type the value of σ_3 has a peak, while σ_1 and σ_2 are smooth as functions of ω. We will discuss these types separately.

It appears that the resonances at which σ_1 has a peak occur at frequencies in the interval $(\omega_2^\sigma, \tilde{\omega}_2^s)$, but away from ω_2^σ. For the frequency range considered in this paper, the value of σ_3 is much smaller than the values of σ_1 and σ_2, except for frequencies very near to ω_2^σ. Hence by (34) the σ_{ij}-matrix, introduced in (19), is at these resonances up to a scalar multiplication-factor approximately of the form:

$$\begin{pmatrix} 1 & -1 \\ -1 & 1 \end{pmatrix}, \tag{72}$$

as can be verified in the Figs. 5–7. Therefore we have at a resonance of this type $Q_1 \approx -Q_2 \gg Q_1 + Q_2$. The σ_{ij} have been computed as functions of ω for three sets of values of the radii of the electrodes. We have chosen $a_1 = 1$, $a_3 - a_2 = 1$ in each set. The values of a_2 and a_3 increase: $a_2 = 2$ in Fig. 5; $a_2 = 4$ in Fig. 6 and $a_2 = 9$ in Fig. 7. It appears that if a_2 increases, then the resonant-frequency decreases. The maximal values of the σ_{ij} in the peak increase for increasing a_2. For the smallest value of a_2, $a_2 = 2$, we could not find with our numerical method a peak for the value of σ_1 at all. When we compute the σ_{ij} matrix using the common practice mentioned above, the error in the computed value of σ_1 is very small, except for frequencies very

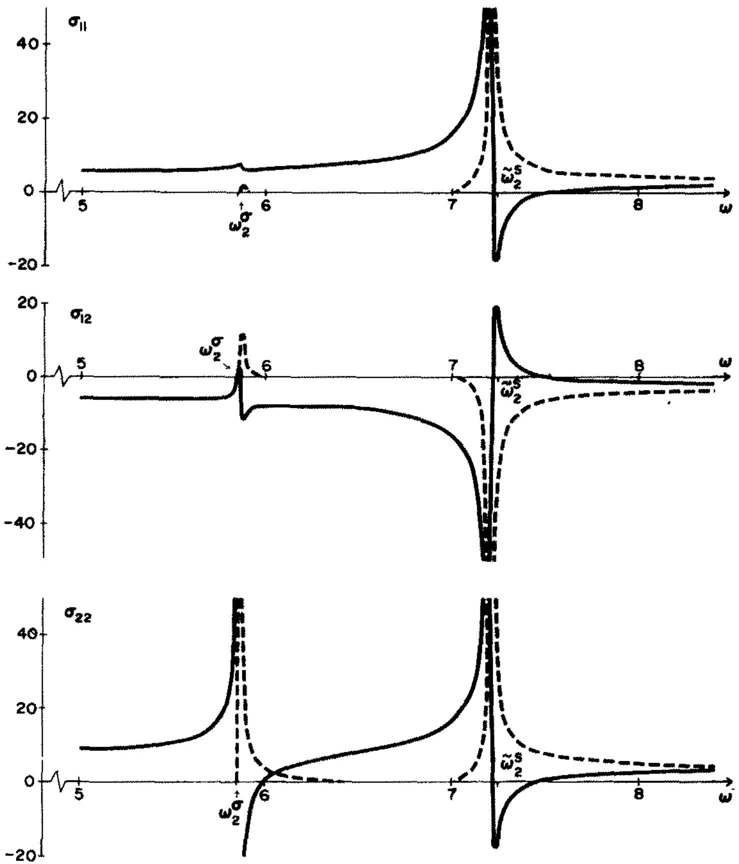

Fig. 6. The σ_{ij}, as functions of ω. Scale-unit is $h\epsilon^{11}$; real part (——); imaginary part (----). Scale-unit for ω is 10^6 rad. sec^{-1}. Radii of the electrodes: $a_1/h = 1$; $a_2/h = 4$; $a_3/h = 5$. The ϵ^0 equals the dielectric constant of vacuum. For dispersion curves and material constants see Fig. 4.

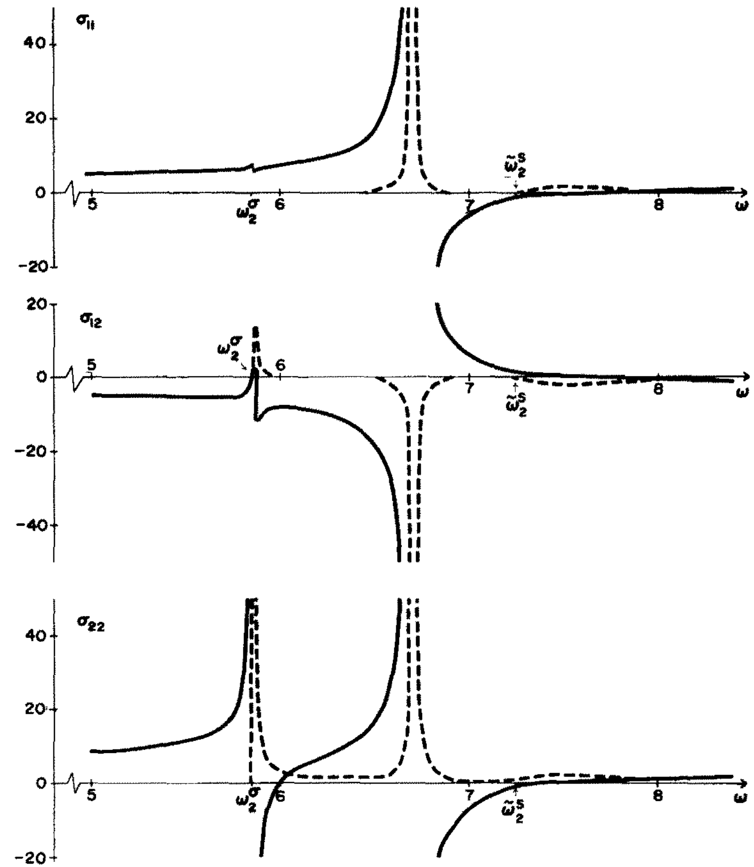

Fig. 7. The σ_{ij}, as functions of ω. Scale-unit is $h\epsilon^{11}$; real part (——); imaginary part, (----). Scale-unit for ω is 10^6 rad. sec^{-1}. Radii of the electrodes: $a_1/h = 1$; $a_2/h = 9$; $a_3/h = 10$. The ϵ^0 equals the dielectric constant of vacuum. For dispersion curves and material constants see Fig. 4.

near to ω_2^σ. This holds also at frequencies where σ_1 has a peak. This may imply that the external electric field is not important for these resonances of the plate.

Near the frequency ω_2^σ the value of σ_3 appears to have a peak for each of the three sets of radii considered. The maximal values of the $|\sigma_{ij}|$ in the peak increase with increasing a_2 and a_3. Since the common practice mentioned above implies $\sigma_3 = 0$, this common practice must not be applied when resonances of this type have to be investigated. This may imply that the external electric field is "indispensible" for these resonances.

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APPENDIX

Derivation of an integral equation and its numerical solution

In order to derive an integral equation for the free charge density q we use the Hankel transformation. For a function $f(r,z)$, its transform is:

$$\bar{f}^\nu(\xi,z) = \int_0^\infty rf(r,z)J_\nu(\xi r)\,dr, \qquad 0 \le \xi < \infty \tag{A1}$$

where J_ν denotes the Bessel function of the first kind of order ν. The inverse transformation is the same transformation:

$$f(r,z) = \int_0^\infty \xi\bar{f}^\nu(\xi,z)J_\nu(\xi r)\,d\xi, \qquad 0 \le r < \infty. \tag{A2}$$

We begin with the range $|z| < 1$. Equation (10) represents three equations. To the first one we apply Hankel transformation with $\nu = 1$ and to the other two we apply it with $\nu = 0$. Then we obtain three ordinary second-order differential equations for $\bar{U}^1(\xi,z)$, $\bar{W}^0(\xi,z)$ and $\bar{V}^0(\xi,z)$ in the range $|z| < 1$. We apply Hankel transformation with $\nu = 1$ to the first equation in matrix eqn (15) and with $\nu = 0$ to the second equation in (15), which gives two equations in $\bar{U}^1$, $\bar{W}^0$ and $\bar{V}^0$ and their first derivatives at $z = 1$ and also at $z = -1$. Now if we give the value of $\bar{V}^0(\xi, z = 1) = \bar{V}^0(\xi, z = -1)$, then $\bar{U}^1$, $\bar{W}^0$ and $\bar{V}^0$ can be computed analytically from these differential equations and boundary conditions. Then the value of D^z can be computed from (3), (4) and (8). It appears that D^z is linear in $V(\xi, z = 1)$, and hence:

$$V(\xi, z = 1) = -K(\xi)D^z(\xi, z = 1 - 0), \tag{A3}$$

where $K(\xi)$ is a function of ξ, ω and the material constants of the plate.

Next we consider the range $|z| > 1$. Applying Hankel transformation with $\nu = 0$ to (8), (11) and (12) and using the condition that V must vanish for $|z| \to \infty$ we obtain a linear relation between $V(\xi, z = 1)$ and $D^z(\xi, z = 1 + 0)$;

$$V(\xi, z = 1) = \frac{1}{\epsilon^0\xi}D^z(\xi, z = 1 + 0). \qquad \xi > 0. \tag{A4}$$

From (A3)–(A4) and Hankel transformation with $\nu = 0$ of (14) we find:

$$V(\xi, z = 1) = \frac{K(\xi)}{\epsilon^0\xi K(\xi) + 1}q(\xi). \tag{A5}$$

Application of the inverse Hankel transformation yields:

$$V(r, z = 1) = \int_0^\infty \frac{\xi K(\xi)}{\epsilon^0\xi K(\xi) + 1}J_0(\xi r)\,d\xi \int_{C_e} sq(s)J_0(\xi s)\,ds. \tag{A6}$$

After a change of the order of integration this can be written as:

$$V(r, z = 1) = \int_{C_e} sG(r,s)q(s)\,ds. \tag{A7}$$

where:

$$G(r,s) = \int_0^\infty \frac{\xi K(\xi)}{\epsilon^0\xi K(\xi) + 1}J_0(\xi r)J_0(\xi s)\,d\xi. \tag{A8}$$

It can be shown (see [4]), that K is a meromorphic function of ξ, and that:

$$K = O(\xi^{-2}) \quad \xi \to 0, \quad K = O(\xi^{-1}) \quad \xi \to \infty, \tag{A9}$$

which implies that the integral in (A8) is convergent for $r \neq s$. At $r = s$ the function G has a logarithmic singularity.

There are poles in the intergrand of (A8). When the damping of the material is neglected, then a finite number of these poles lies on the path of integration. The real ξ-values at which they occur are given as a function of ω by the dispersion curves for a plate without electrodes discussed in Section 7. The value of the integral is computed as the sum of a Cauchy principal value and $\pm\pi i$ times the residu of the pole, where the sign is determined using the limited absorption principle (see [4]). The contribution of each pole to the value of $G(r, s)$ is $O(r^{-1/2})$ for $r \to \infty$. It represents a wave which transports energy through the plate to infinity and hence induces a "radiation resistance". The effect of the poles at complex ξ-values, which are given by the complex dispersioncurves, appears in the numerically computed value of the integral in (A8).

When the electric potential is prescribed on the electrodes we obtain from (A7) an integralequation of the first kind for q:

$$\int_{C_e} SG(r, s)q(s)\,ds = V_1, \quad 0 \leq r \leq a_1$$
$$= V_2, \quad a_2 \leq r \leq a_3 \tag{A10}$$

This equation is solved numerically as follows: We choose $n + m$ collocation-points r_k on C_e:

$$0 < r_1 < r_2 \ldots < r_n = a_1 < a_2 = r_{n+1} < \ldots < r_{n+m} = a_3 \tag{A11}$$

and we introduce linear interpolation functions $f_k(s)$, which have the value one or zero at the collocation points (see Fig. 3). The function f_1 is chosen in a special way, due to the axisymmetry. We approximate the solution q of (A10) by an expression of the form:

$$q(s) = \sum_{k=1}^{n+m} q_k f_k(s) R(s), \qquad s \epsilon C_e \tag{A12}$$

where:

$$R(s) = \{|a_1 - s| \cdot |a_2 - s| \cdot |a_3 - s|\}^{-1/2} \tag{A13}$$

The q_k are coefficients and the function $R(s)$ is introduced, since we expect by virtue of the logarithmic singularity of G at $r = s$ the free-charge density to have a square-root singularity at the edge of the electrodes. Collocation at the points r_k gives $n + m$ equations for q_k:

$$\sum_{k=1}^{n+m} A_{lk} q_k = V_1, \qquad l = 1, \ldots, n \tag{A14}$$

where:

$$A_{lk} = \int_{C_e} sG(r_l, s) f_k(s) R(s)\,ds. \tag{A15}$$

The A_{lk} are computed numerically and then the q_k follow from (A14). The values of Q_1 and Q_2 follow from a numerical integration of (A12). The values of the σ_{ij} follow from (19) by computing the Q_i for two different sets of values of V_1 and V_2.